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Nonlinear Landau damping in a relativistic plasma

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A generalized non-local nonlinear Schrödinger equation describing the phenomenon of nonlinear Landau damping in a relativistic two-component plasma is deduced using the kinetic-theory approach of Vlasov. Parameters appearing in the equation are evaluated explicitly for the case of a Maxwellian distribution function.

1. Introduction

The phenomenon of Landau damping occupies a central position in the study of many important properties of plasmas (Davidson 1992). In general, there are two kinds of Landau damping: linear and nonlinear (Weiland, Ichikawa & Wilhelmsson 1977; Weiland 1981; Mjølhus & Wyller 1986). Perhaps the first attempt to incorporate the effects of Landau damping in the study of solitons was that of Ott & Sudan (1969, 1970), who derived a nonlinear dispersion relation containing a term representing Landau damping. Several attempts have since been made to study the influence of this important phenomenon in other situations, such as in relativistic collisionless plasmas (Roy Chowdhury, Pakira & Paul 1988), magnetized plasmas (Som *et al.* 1992) and two-temperature systems (Chen (1984). Landau damping has been discussed in detail by Chen (1984).

Ichikawa, Imamura & Taniuti (1972) studied the effect of resonant particles having group velocity $v_g = \partial\omega/\partial K$ on finite-amplitude plasma waves. They showed that such waves can be described by a nonlinear Schrödinger equation with an extra nonlinear non-local term that actually represents nonlinear Landau damping. A Lagrangian formalism for nonlinear wave packets was adopted by Dewar (1972), who determined a non-local nonlinear frequency shift and obtained a similar equation. All of these considerations were non-relativistic. Therefore here we consider a collisionless isotropic plasma consisting of electrons and ions, with the electrons being assumed to be relativistic. The whole system is described in the framework of Vlasov kinetic theory (Kadomtsev 1965; Baryakhtar *et al.* 1992). We have then followed the approach of Fried & Ichikawa (1973) to investigate the nonlinear dispersion relation, and hence have obtained a nonlinear Schröinger equation with an extra nonlinear non-local term. As mentioned above, this term represents nonlinear Landau damping in the relativistic plasma.

2. Formulation

Let f_e and f_i denote the velocity distribution functions for the electrons and ions in the plasma, with the electrons assumed to be relativistic. With ϕ denoting the electrostatic potential, the Vlasov kinetic equations describing a collisionless two-component magnetized plasma are

$$\left.\begin{aligned}
&\frac{\partial f_e}{\partial t}+v_e\frac{\partial f_e}{\partial x}e\frac{\partial \phi}{\partial x}\frac{\partial f_e}{\partial p_e}=0,\\
&\frac{\partial f_i}{\partial t}+v_i\frac{\partial f_i}{\partial x}-\frac{e}{m_i}\frac{\partial \phi}{\partial x}\frac{\partial f_i}{\partial v_i}=0,\\
&\frac{\partial^2 \phi}{\partial x^2}=-4\pi e\int (f_i-f_e)\,dp.
\end{aligned}\right\}\tag{1}$$

We denote the mass of the electron by $m_e' = m_e\,(1-v_e^2/c^2)^{-\frac{1}{2}}$, m_e being the rest mass. For the ions $m_i' = m_i$, since they are taken to be non-relativistic. Converting all the momentum variables to the corresponding velocities, we can rewrite the above set of equations as

$$\left.\begin{aligned}
&\frac{\partial f_e}{\partial t}+v_e\frac{\partial f_e}{\partial x}+\frac{e}{m_e}\chi(v_e)\frac{\partial \phi}{\partial x}\frac{\partial f_e}{\partial v_e}=0,\\
&\frac{\partial f_i}{\partial t}+v_i\frac{\partial f_i}{\partial x}-\frac{e}{m_i}\frac{\partial \phi}{\partial x}\frac{\partial f_i}{\partial v_i}=0,\\
&\frac{\partial^2 \phi}{\partial x^2}=-4\pi e m_i\int f_i\,dv_i+4\pi e m_e\int f_e\,\eta(v_e)\,dv_e,
\end{aligned}\right\}\tag{2}$$

where

$$\eta(v_e)=1+\frac{3}{2}\frac{v_e^2}{C^2},$$

$$\chi(v_e)=\frac{1-v_e^2/2c^2}{1+v_e^2/c^2}.$$

To derive the dispersion relation, we set

$$\left.\begin{aligned}
&f_e=F_1(v_e)+f_{ei},\\
&f_i=F_2(v_i)+f_{i1},\\
&\phi=\phi_1,
\end{aligned}\right\}\tag{3}$$

where F_1 and F_2 are the equilibrium values of the corresponding distributions, and f_{e1}, f_{i1} and ϕ_1 are perturbations, each of which is proportional to $\exp(iKX-i\omega t)$. Substitution of (3) into (2) leads to

$$\left.\begin{aligned}
&f_{e1}=-\frac{e\chi(v_e)F_1'(v_e)}{m_e(v_e-\omega/K)}\phi_1\\
&f_{i1}=\frac{eF_2'(v_i)}{m_i(v_i-\omega/K)}\phi_1
\end{aligned}\right\}\tag{4}$$

and
$$1 = 4\pi e^2 \int \frac{F_2'(v_i)\,dv_i}{K^2(v_i-\omega/K)} + 4\pi e^2 \int \frac{\eta(v_e)\,\chi(v_e)}{K^2(v_e-\omega/K)} F_1'(v_e)\,dv_e, \tag{5}$$

which will be written as
$$\epsilon(K,\omega,0) = 0$$

and is the linear dispersion relation not involving the amplitude. A method of computing corrections to (5) was given by Fried & Ichikawa (1973), who showed how a nonlinear Schrödinger equation can be deduced from it. In the following section we shall briefly discuss this method.

3. Derivation of the NLS equation

The basic idea behind the analysis of Fried & Ichikawa (1973) is to consider the form of the wave at time $t = 0$ to be

$$\phi(X,0) = \int dK\,\phi_K\, e^{iKX} + \text{C.C.}, \tag{6}$$

where ϕ_K is peaked around $K = K_0$ (C.C. stands for complex conjugate). In linear theory the long-time behaviour of the system is given by

$$\phi(x,t) = \int dK\,\phi_K\, e^{i(Kx-\omega t)} + \text{C.C.} \tag{7}$$

where $\omega = \omega(K)$ is the least-damped solution of the dispersion relation.

Here we shall consider the case where both the amplitude and the spread of K values are small, so that an expansion of the dispersion relation, first in A^2 and then in $K-K_0$, is appropriate; thus

$$(K,\omega,A) = \epsilon(K,\omega,0) + A^2\frac{\partial\epsilon}{\partial A^2} + \ldots = 0. \tag{8}$$

For ω, we get
$$\omega = \Omega(K) + A^2\frac{\partial\omega}{\partial A^2} + \ldots, \tag{9}$$

where $\Omega(K)$ is a solution of the linearized dispersion relation

$$\epsilon[K,\Omega(K),0] = 0. \tag{10}$$

Again,
$$\Omega(K) = \Omega(K_0+K-K_0) = \omega_0 + v_g\mathscr{K} + \frac{v_g}{2}\mathscr{K}^2 + \ldots, \tag{11}$$

with $\omega_0 = \Omega(K_0)$, $v_g = \Omega'(K_0)$, $v_g' = \Omega''(K_0)$ and $\mathscr{K} = K-K_0$, where the prime denotes the derivative with respect to K. Thus we get

$$\left.\begin{aligned} \omega &= \omega_0 + \nu \\ \nu &= v_g\mathscr{K} + \tfrac{1}{2}v_g\mathscr{K}^2 + \mu A^2, \end{aligned}\right\} \tag{12}$$

with
$$\mu = \frac{\partial\omega}{\partial A^2} = -\frac{\partial\epsilon}{\partial A^2}\Big/\frac{\partial\epsilon}{\partial\omega}.$$

Substituting (6) into (7), we obtain

$$\phi(x,t) = \int d\mathscr{K}\, \phi_{\mathscr{K}+K_0}\, e^{i[(\mathscr{K}+K_0)\,x+(\omega_0+\nu)\,t]} + \mathrm{C.C.}$$

$$= \psi(x,t)\, e^{i(K_0 x-\omega_0 t)} + \mathrm{C.C.}, \tag{13}$$

$$\psi(x,t) = \int d\mathscr{K}\, \phi_{\mathscr{K}+K_0}\, e^{i(\mathscr{K}x-\nu t)}. \tag{14}$$

From (12) and (14), we can immediately infer that ψ satisfies

$$i\left(\frac{\partial\psi}{\partial t}+v_g\frac{\partial\psi}{\partial x}\right)+\frac{v_g'}{2}\frac{\partial^2\psi}{\partial x^2}-\mu\,|\psi|^2\psi = 0, \tag{15}$$

which is the usual nonlinear Schrödinger equation, when we have identified $|\psi|$ as the amplitude A. However, the dependence of ϵ on A^2 may be of a more general form

$$\int dx'\, S(x-x')\, A^2(x'),$$

where $S(x)$ is a kernel, whence

$$\epsilon(K,\omega A) = \epsilon(K,\omega,0) + \int dx'\, S(x-x')\, A^2(x') + \dots\quad .$$

Now,

$$\partial\epsilon/\partial A^2 = \int dX'\, S(x-x'),$$

so that

$$\mu = -\int dx'\, S(x-x')\left(\frac{\partial\epsilon}{\partial\omega}\right)^{-1},$$

and

$$\nu = v_g\,\mathscr{K} + \tfrac{1}{2}v_g'\,\mathscr{K}^2 - \int dx'\, S(x-x')\left(\frac{\partial\epsilon}{\partial\omega}\right)^{-1} A^2(X'). \tag{16}$$

Thus we get a nonlinear non-local Schröinger equation

$$i\psi_t + p\psi_{xx} + \psi\int dx'\, Q(x-x')\,|\psi(x')|^2 = 0. \tag{17}$$

In the following we shall first evaluate the correction to the dispersion relation (5) due to the nonlinear effect of the amplitude. Then the analysis described above will immediately lead to our desired result.

4. Correction to the linear dispersion relation

In the present case of a relativistic plasma, we go back to (2). To obtain the desired nonlinear correction to (5), we take spatial Fourier transforms of f_e, f_i and ϕ and Laplace transforms in time (Roy Chowdhury & Roy Chowdhury 1991; Chakravorty, Pakira & Roy Chowdhury 1993):

$$f_e(K,t,v_e) = \int f_e(x,t,v_e)\, e^{-iKx}\, dx$$

$$f_e(K,S,v_e) = \int_0^{\alpha} f_e(K,t,v_e)\, e^{-st}\, dt, \quad \operatorname{Re} S > S_0, \tag{18}$$

the Laplace inversion being

$$f_e(k,t,v_e) = \frac{1}{2\pi i}\int_{s_0-i\alpha}^{s_0+i\alpha} f_e(K,s,v_e)\, e^{st}\, dS, \tag{19}$$

where it is understood that an appropriate semicircular contour has been chosen. Similar formulae also hold for f_i and ϕ.

We can now rewrite (19) in a more compact and useful way. We set $S = -i\omega$; then

$$f_e(x,v_e,t) = \frac{1}{(2\pi)^2}\int_{\omega_0-i\alpha}^{\omega_0+i\alpha} d\omega \int dK f_e(K,\omega,v_e)\, e^{i(kx-\omega t)}, \tag{20}$$

so, instead of (3), we now substitute

$$f_e(\chi,v_e,t) = F_1(v_e) + \int f_e(p,v_e)\, e^{i(Kx-\omega t)}\, dp, \tag{21}$$

where p stands for (K,ω) and $dp = -dK\, d\omega/(2\pi)^2$. Similarly, we set

$$\begin{aligned} f_i(x,v_i,t &= F_2(v_i) + \int f_i(p,v_i)\, e^{i(K_x-\omega t)}\, dp, \\ \phi(x,t) &= \int dp\, e^{i(Kx-\omega t)}\, \phi(p). \end{aligned} \tag{22}$$

Equation (??) yields,

$$\begin{aligned} f_e(p,v_e) = (\omega-kv_e)^{-1}\Bigg\{ & if_e(K,v_e,t=0) + \frac{e}{m_e}K\chi(v_e)\,\phi(p)\,F_1'(v_e) \\ & + \frac{e}{m_e}\int dp'\, K'\phi(p')\, D_1\Bigg[[\omega-\omega'-(K-K')\,v_e]^{-1}\Bigg(if_e(K,v_e,t=0) \\ & + \frac{e}{m_e}\chi(v_e)\,(K-K')\,\phi(p-p')\,F_1'(v_e) \\ & + \frac{e}{m_e}\int dp''\phi(p'')\,K''D_1 f_e(p-p'-p'',v_e)\Bigg)\Bigg]\Bigg\}. \end{aligned} \tag{23}$$

In the Laplace convolution integral we choose the ω contours to lie above all singularities of $\phi(K,\omega)$, and the $\omega-\omega'$ contour to lie above all singularities of $f(K,v,\omega-\omega')$. Iteration of (23) leads to

$$\begin{aligned} f_e(p,v_e) = I_1(p,v_e) + \frac{e}{m_e}(\omega-Kv_e)^{-1}\Bigg\{ & K\chi(v_e)\,\phi(p) \\ & + \frac{e}{m_e}\int dp'\, K'\phi(p')\, D_1\{(K-K')\,\phi(p-p')\,\chi(v_e)\,[(\omega-\omega')-(K-K')\,v_e]^{-1}\} \\ & + \left(\frac{e}{m_e}\right)^2 \int dp'\, dp''\, K'\phi(p')\, D_1\{[(\omega-\omega')-(K-K')\,v_e]^{-1}K''\phi(p'')\} \\ & \times D_1\{\chi(v_e)\,(K-K'-K'')\,\phi(p-p'-p'')\,[(\omega-\omega'-\omega'') \\ & -(K-K'-K'')\,v_e]^{-1}\Bigg\} F_1'(v_e) \end{aligned} \tag{24}$$

where $I_1(p, v_e)$ consists of all the inhomogeneous terms, namely those linear in the initial perturbations, and each operator D_1 acts at all functions of v_e to its right. A similar analysis of the ion and Laplace equations yields

$$\begin{aligned} f_i(p,v_i) = I_2(p,v_i) - \frac{e}{m_i}(\omega - Kv_i)^{-1}\Big\{ & K\phi(p) \\ & - \frac{e}{m_i}\int dp'\,K'\phi(p')\,D_2\{[\omega-\omega'-(K-K')\,v_i]^{-1}\,(K-K')\,\phi(p-p')\} \\ & + \left(\frac{e}{m_i}\right)^2 \int dp'\,dp''\,K'\phi(p')\,D_2\{[\omega-\omega'-(K-K')\,v_i]^{-1}K''\phi(p'')\}\,D_2 \\ & \times\{(K-K'-K'')\,\phi(p-p'-p'')\,[(\omega-\omega-\omega)-(K-K'-K'')\,v_i]^{-1}\}\Big\}\,F_2'(v_i), \end{aligned} \tag{25}$$

$$-K^2\phi(p) = -4\pi e m_i \int f_i(p,v_i)\,dv_i + 4\pi e m_e \int \eta(v_e)\,p_e(p,v_e)\,dv_e. \tag{26}$$

With the help of (24)–(26), we get

$$\begin{aligned} -K^2\epsilon(p)\,\phi(p) = \phi_0(p) & + \int dp'\,[m_i M_{11}(p,p') + m_e M_{21}(p,p')]\,\phi(p')\,\phi(p-p') \\ & + \int dp'\,dp''\,[m_i M_{12}(p,p',p'') \\ & + m_e M_{22}(p,p',p'')]\,\phi(p')\,\phi(p'')\,\phi(p-p'-p''), \end{aligned} \tag{27}$$

where

$$\begin{aligned} \epsilon(p) = \frac{1}{4\pi_e}\Bigg[& 1 - 4\pi e^2 \int \frac{F_2'(v_i)\,dv_i}{K^2(v_i-\omega/K)} \\ & - 4\pi e^2 \int \frac{\chi(v_e)\,\eta(v_e)\,F_1'(v_e)}{K^2(v_e-\omega/K)}\,dv_e\Bigg], \end{aligned} \tag{28}$$

$$\phi_0(p) = m_e \int \eta(v_e)\,I_1(p,v_e)\,dv_e - m_i \int I_2(p,v_i)\,dv_i. \tag{29}$$

In (27) and (29) the symbols M_{11}, M_{12}, M_{21} and M_{22} stand for the following complicated expressions involving the initial distribution functions f and x, η:

$$M_{11}(p,p') = \left(\frac{e}{m_i}\right)^2 KK'(K-K') \int \frac{dv_i\,F_2'(v_i)\,(\omega-kv_i)^{-2}}{(\omega-\omega')-(K-K')\,v'}, \tag{30a}$$

$$M_{21}(p,p') = -\left(\frac{e}{m_e}\right)^2 K'(K-K') \int \left[\frac{\eta(v_e)\,K}{(\omega-Kv_e)^2} + \frac{\eta'(v_e)}{\omega-KV_e}\right] F_1'\,\chi(v_e)\,dV_e, \tag{30b}$$

$$\begin{aligned} M_{12}(p,p',p'') = & -\left(\frac{e}{m_i}\right)^3 KK'K''(K-K'-K'') \\ & \times \int \frac{dv_i(\omega-Kv_i)^{-2}D_2F_2'(v_i)}{[\omega-\omega'-(K-K')\,v_i]\,[(\omega-\omega'-\omega''-(K-K'-K'')\,v_i]}, \end{aligned} \tag{30c}$$

$$\begin{aligned} M_{22}(p,p',p'') = & -\left(\frac{e}{m_e}\right)^3 K'K''(K-K'-K'') \int dv_e \left[\frac{K\eta(v_e)}{(\omega-Kv_e)^2} + \frac{\eta'(v_e)}{\omega-Kv_e}\right] \\ & \times \frac{D_1\{\chi(v_e)\,F_1'(v_e)\}}{[(\omega-\omega')-(K-K')\,v_e]\,[(\omega-\omega'-\omega'')-(K-K'-K'')\,v_e]} \end{aligned} \tag{30d}$$

Equation (27) is now used to find the nonlinear frequency shift. Since this equation cannot be solved exactly, we again make use of perturbation theory. Let us consider the iterative solution of (27) in the following form, up to terms of third order:

$$\phi^{(2)}(p) = -\int dp' \, [m_i M_{11}(p,p') + m_e M_{21}(p,p')] \frac{\phi^{(1)}(p')\,\phi^{(1)}(p-p')}{K^2\epsilon(p)}, \tag{31}$$

$$\begin{aligned} -K^2\epsilon(p)\,\phi^{(3)}(p) = & \int dp' \, [m_i M_{11}(p,p') + m_e M_{21}(p,p')]\,\phi^{(2)}(p')\,\phi^{(1)}(p-p') \\ & + \int dp' [m_i M_{11}(p,p') + m_e M_{21}(p,p')]\,\phi^{(1)}(p')\,\phi^{(2)}(p-p') \\ & + \int dp' \, dp'' \, [m_i M_{12}(p,p',p'') \\ & + m_e M_{22}(p,p',p'')]\phi^{(1)}(p')\,\phi^{(1)}(p'')\,\phi^{(1)}(p-p'-p''). \end{aligned} \tag{32}$$

We are interested in the case where $\phi(K,\omega)$ has a peak value. Let $\phi(K,\omega)$ be peaked around $K = \pm K_0$; then we can set

$$\phi(K,\omega) = \frac{i\hat{\phi}(K)}{\omega - \Omega(K)}, \tag{33}$$

where $\mathrm{Im}\,[\omega - \Omega(K)] > 0$. For the above iteration process, we take

$$\phi^{(1)} = \frac{i\hat{\phi}(K)}{\omega - \Omega(K)}. \tag{34}$$

It is then not difficult to see that $\phi(2)$ is zero: if we demand that $\phi^{(1)}$ and its iterated values should all be peaked near $K = \pm K_0$ then, from (31), we find that all three moduli $|p|$, $|p'|$ and $|p-p'|$ should be equal to K_0 or $-K_0$ simultaneously, which is impossible. The next finite contribution comes from the third-order iteration $\phi^{(3)}$.

We now choose the ω' and ω'' contours in (32) to be infinite semicircle in the lower half-plane; we then enclose the poles at $\omega' = \Omega(K') \equiv \Omega'$ and $\omega'' = \Omega(K'') \equiv \Omega''$ coming from the factors of $\phi^{(1)}(p)$, and those at $\omega' = K'v_i$ of M_{11} $(p,\ p-p')$ and $\omega' = K'v_e$ of $M_{21}(p,\ p-p')$. From (32), we immediately get

$$\begin{aligned} -K^2\epsilon(p)\,\phi^{(3)}(p) = (2\pi)^{-2}\, i \int & d(K',\Omega')\, d(K'',\Omega'') \Big\{ [m_i M_{12}(K,\Omega;K',\Omega';K'',\Omega'') \\ & + m_e M_{22}(K,\Omega;K',\Omega';K'',\Omega'')] \\ & - \{[m;M_{11}(K,\Omega;K-K',\Omega-\Omega') + m_e M_{21}(K,\Omega;K-K',\Omega-\Omega')] \\ & - [m_i M_{11}(K,\Omega;K',\Omega') + m_e M_{21}(K,\Omega;K',\Omega')]\} \\ & \times [m_i M_{11}(K-K',\Omega-\Omega';K'',\Omega'') + m_e M_{21}(K-K',\Omega-\Omega';K'',\Omega'')] \Big\} \\ & \times (K-K')^{-2}\,\epsilon(p-p')^{-1}\,\phi^{(1)}(K',\Omega')\,\phi^{(1)}(K'',\Omega'')\,\phi^{(1)}(K-K'-K'',\Omega-\Omega'-\Omega''). \end{aligned} \tag{35}$$

We now define

$$\left.\begin{aligned} \epsilon(K) &= \epsilon(K, \Omega(K)), \\ R_{12}(K,K',K'') &= M_{12}(K,\Omega;K',\Omega';K'',\Omega''), \\ R_{22}(K,K',K'') &= M_{22}(K,\Omega;K',\Omega';K'',\Omega''), \\ R_{11}(K,K') &= M_{11}(K,\Omega;K',\Omega'), \\ R_{21}(K,K') &= M_{21}(K,\Omega;K',\Omega'). \end{aligned}\right\} \tag{36}$$

Using these, (35) can be rewritten more compactly as

$$\begin{aligned} &-K^2\epsilon(p)\,\phi^{(3)}(p) \\ &\quad = (2\pi)^{-2}\,i\int \alpha K'\,dK''\,R(K,K',K'')\,\hat{\phi}(K')\,\hat{\phi}(k'')\,\hat{\phi}(K-K'-K''), \end{aligned} \tag{37}$$

where the kernel $R(K,K',K'')$ has the form

$$\begin{aligned} R(K,K',K'') &= m_i R_{12}(K,K',K'') + m_e R_{22}(K,K',K'') \\ &\quad -[m_i R_{11}(K,K-K') + m_e R_{21}(K,K-K') \\ &\quad + m_i R_{11}(K,K') + m_e R_{21}(K,K')]\,[m_i R_{11}(k-K',K'') \\ &\quad + m_e R_{21}(K-K';K'')]\,(K-K)^{-2}[\epsilon(K-K')]^{-1}\,[\omega-\Omega'-\Omega''-\Omega(K-K'-K'')]^{-1} \\ &\quad + T(K,K',K''). \end{aligned} \tag{38}$$

The term $T(K,K',K'')$ is

$$\begin{aligned} T(K,K',K'') &= -KK'(K-K')\,[m_i M_{11}(K-K',\omega-K'v_i;K'',\Omega'') \\ &\quad + m_e M_{21}(K-K',\omega-K'v_e;K'',\Omega'')] \\ &\quad \times\left\{\frac{1}{m_i}\int\frac{dv_i F_2'}{(\omega-Kv_i)^2\,[\omega-K'v_i-\Omega''-\Omega(K-K'-K'')]}\right. \\ &\quad \times(K'v_i-\Omega'-i\eta_1)^{-1}\,[\epsilon(K-K',\omega-K'v_i)]^{-1} \\ &\quad -\frac{1}{m_e}\int dv_e F_1'\,\chi(v_e)\left[\frac{\eta(v_e)K}{(\omega-Kv_e)^2}+\frac{\eta'(v_e)}{(\omega-Kv_e)}\right](\omega-K'v_e-\Omega''-\Omega(K-K'-K''))]^{-1} \\ &\quad \left.\times(K'v_e-\Omega'+i\eta_2)^{-1}\,[\epsilon(K-K';\omega-K'v_e)]^{-1}\right\}(K-K)^{-2}, \quad \eta_1,\eta_2>0 \end{aligned} \tag{39}$$

This term actually comes from the poles at $\omega' = K'v_i$ and $\omega' = K'v_e$ in M_{11} and M_{21} respectively, but makes no contribution to the final result since it has no factor $(\omega-\Omega)^{-1}$.

We now express K, K', and K'' in terms of small deviations q, q' and q'' from K_0. We write

$$\begin{aligned} R_{11}(K,K') &= R_{11}(K,K_0+q') \\ &= M_{11}(K,\Omega;K_0+q',\Omega(K_0+q')), \quad \text{etc.} \end{aligned} \tag{40}$$

Since $\phi(K,\omega)$ has a peak near $K \approx \pm K_0$, we can assume that the initial perturbations $f_e(K,v_e,t=0)$ and $f_i(K,v_i,t=0)$ are non-zero only for $K \approx \pm K_0$. We then assume that the wavepacket can be written as

$$\phi(x,t) = \psi(x,t)\,e^{i(K_0x-\omega_0t)} + \text{C.C.},$$

with

$$\psi(x,t) = (2\pi)^{-1}\int dq\,\psi(q)\,e^{i(qx-\nu t)}, \tag{41a}$$

with $\nu = \Omega(K_0+q)-\Omega(K_0)$, whence, from (35), we get

$$-K^2\phi^{(3)}(p)\,\epsilon(p) = (2\pi)^{-2}\,i\int dq'\,dq''$$

$$\times S(q,q',q'')\,\psi(q')\,\psi^*(-q'')\,\psi(q-q'-q'')\,[\omega-\Omega(K)]^{-1},$$

$$S(q,q',q'') = R(K_0+q,K_0+q',\,-K_0+q'')+R(K_0+q,K_0+q',K_0+q'')$$

$$+R(K_0+q,\,-K_0+q',K_0+q''). \quad (41b)$$

The explicit form of R can easily be ascertained from (37). The expression is very long, and we shall only give here a term representative of its structure. For example,

$$R_{12}(K_0+q,K_0+q;K) = -\left(\frac{e}{m_i}\right)^3 (K_0+q)\,(K_0+q')\,K''(q-q'-q'')$$

$$\times\int dv_i\,[\Omega(K_0+q)-Kv_i]^{-2}\,[(\omega-\omega')-(q-q')\,v_i]^{-1}\,D_2F_2'(v_i), \quad (42)$$

with similar expressions for $R_{12}(K_0+q,K_0+q',K'')$, $R_{22}(K_0+q,K_0+q',K'')$ etc. Setting $q = q' = q'' = 0$ in all the terms without poles, we can at once obtain simplified expressions for them.

There are many terms with $\omega-kv_i$ and $\omega-Kv_e$ in the denominator, which are handled by the usual Landau damping procedure. For this, let us consider the term R_{11}:

$$R_{11}(K,K') = \left(\frac{e}{m_i}\right)^2 KK'(K-K')\int dv_i F_2'\,[\omega-\Omega'-(K-K')\,v_i]^{-1}(\omega-Kv_i)^{-2}, \quad (43)$$

$$\lim_{K'\to K}\,(K-K')\int dv\,h(v)/[(K-K')\,v-(\omega-\Omega')]$$

$$= P\int dv\frac{h(v)}{v-v_g}+\operatorname{sgn}(K-K')\,ih(v_g) \quad (44)$$

with $\omega = \Omega+i\eta$, $\eta\to 0_+$, and $\operatorname{sgn}(x)\equiv x/|x|$. So we can let $K,K'\to K_0$ in R_{11}, but the result is still a function of $K'-K$, with

$$K = K_0+q,\quad K' = K_0+q',\quad (q,q',q-q',q'')\to 0.$$

Finally, we obtain

$$R_{11}(K,K')\to\omega_{11}(K_0,q-q'),$$

with

$$\omega_{11} = -\left(\frac{e}{m_i}\right)^2\int\frac{dv_i F_2'}{(v_i-v_g-i\eta/q)\,(v_i-u)^2}$$

$$= \omega_1(K)+i\operatorname{sgn}(q)\,\omega_2(K), \quad (45)$$

$$\omega_1 = -\left(\frac{e}{m_i}\right)^2 P\int dv_i\frac{F_2'}{(v_i-u)^2(v_i-v_g)},$$

$$\omega_2 = \pi\frac{F_2'(v_g)}{(v_g-u)^2}.$$

Similar contributions from other terms lead to

$$R_{21}(K,K') \to \omega_{21}(K_0, q-q')$$
$$= \omega_3(K) + i\,\mathrm{sgn}\,(q)\,\omega_4(K),$$
$$\omega_3(K) = \left(\frac{e}{m_e}\right)^2 P\left[\int dv_e \frac{F_1'(v_e)\,\chi(v_e)\,\eta(v_e)}{(v_e-v_g)\,(v_e-u)^2} + \int dv_e \frac{F_1'(v_e)\,\eta'(v_e)\,\chi(v_e)}{(v_e-v_g)\,(v_e-u)}\right], \tag{46}$$

$$\omega_4(K) = \left(\frac{e}{m_e}\right)^2 \pi\,[F_1'(v_g)\,\eta(v_g)\,\chi(v_g)\,(v_g-u)^{-2} + F_1'(v_g)\,\eta'(v_g)\,\chi(v_g)\,(v_g-u)^{-1}],$$

$$R_{12}(K, K_0+q', K_0+q'') + R_{12}(K, K_0+q', -K_0+q''') \to C_{11}(K_0, q-q), \tag{47}$$

where
$$C_{11}(K,q) = C_1 + iC_2\,\mathrm{sgn}\,(q),$$
$$C_1 = -\left(\frac{e}{m_i}\right)^3 P\int dv_i\,(v_i-u)^{-2}\,(v_i-v_g)^{-1} D_2 F_2'(v_i-v_g)\,(v_i-u)^{-2},$$
$$C_2 = -\left(\frac{e}{m_i}\right)^3 \pi F_2'(v_g)\,(v_g-u)^{-4},$$
$$R_{22}(K, K_0+q', K_0+q'') + R_{22}(K, K_0+q', -K_0+q''') \to C_{22}(K_0, q-q'),$$
$$C_{22}(K,q) = C_3 + iC_4\,\mathrm{sgn}\,(q),$$
$$C_3 = -\left(\frac{e}{m_e}\right)^2 P\left\{\int dv_e\,\eta(v_e-v_g)\,(v_e-u)^{-2}(v_e-v_g)^{-1} \times D_1\,[\chi(v_e-v_g)F_1'(v_e-v_g)]\,(v_e-u)^{-2} + \int dv_e\,\eta'(v_e-u_g)\,(v_e-u)^{-1}(v_e-v_g)^{-1} \times D_1[\chi(v_e-v_g)F_1'(v_e-v_g)]\,(v-u)^{-1}\right\}, \tag{48}$$

$$C_4 = -\left(\frac{e}{m_e}\right)^3 \pi\,[\chi(v_g)F_1'(v_g)\,(v_g-u)^{-4} + \chi(v_g)F_1(v_g)\,(v_g-u)^{-2}]. \tag{49}$$

Furthermore,

$$R_{11}(K-K', K_0+q'') + R_{11}(K-K', -K_0+q''') \to \omega_{11}(K_0, q-q'), \tag{50}$$

where ω_{11} is given by (45). We also observe that

$$R_{21}(K-K', K_0+q'') + R_{11}(K-K', -K_0+q''') \to \omega_{21}(K_0, q-q')$$

where ω_{21} is given by (46). A contribution to the singularity in $K-K'$ also comes from $(K-K')^2\,\epsilon(K-K')$. We have $\lim\,(K-K')^2\,\epsilon(K-K') = -\Delta(K, q-q)$,

where
$$\Delta(K,q) = \int dv_i F_2'(v_i)\left(v_i - v_g - \frac{i\eta_1}{q}\right)^{-1} + \int dv_e F_1'(u_e)\left(v_e - v_g - \frac{i\eta_2}{q}\right)^{-1}$$
$$= \Delta_1(K) + i\,\mathrm{sgn}\,(q)\,\Delta_2(K),$$
$$\Delta_1(K) = P\left[\int dv_i F_2'(v_i)\,(v_i - v_g)^{-1} + \int dv_e F_1'(v_e)\,(v_e - v_g)^{-1}\right], \tag{51}$$
$$\Delta_2(K) = \pi\,[F_2'(v_g) + F_1'(v_g)]. \tag{52}$$

5. Contribution of non-singular terms

The contributions from terms without any singularities can easily be evaluated by setting $q' = q'' = 0$ and $K = K_0$ in the expression (41b) for $S(q,q',q'')$. We have

$$R_{12}(K,K',K'') = -\left(\frac{e}{m_i}\right)^3 KK'K''(K-K'-K'')\int dv_i\,(\omega - Kv_i)^{-2}$$
$$\times[\omega - \omega' - (K-K')\,v_i]^{-2}$$
$$\times D_2 F_2'(v_i)\,[(\omega-\omega'-\omega'') - (k-K'-K'')\,v_i]^{-1}, \tag{53}$$
$$R_{12}(K,\,-K,K) = \left(\frac{e}{m_i}\right)^3 \frac{1}{2}\int dv_i\,(v_i-u)^{-3} D_2 F_2(v_i)\,(v_i-u)^{-1}, \tag{54}$$
$$= \tfrac{1}{2}B \quad \text{(say)}.$$

Similarly, from (30d),

$$R_{22}(K,\,-K,K) = \left(\frac{e}{m_e}\right)^3 \frac{1}{2}\left\{\int dv_e\,\eta(v_e)\,(v_e-u)^{-3} D_1\,[\chi(v_e)F_1'(v_e)\,(v_e-u)^{-1}]\right.$$
$$\left.\times\, dv_e\,\eta'(v_e)\,(v_e-u)^{-2} D_1\,[\chi(v_e)F_1'(v_e)\,(v_e-u)^{-1}\}\right\} = -\tfrac{1}{2}D \quad \text{(say)}, \tag{55}$$
$$R_{11}(K,\,-K) = \left(\frac{e}{m_i}\right)^2 \int dv_i F_2'(v_i)\,(v_i-u)^{-3} = -A \quad \text{(say)},$$
$$R_{21}(K,\,-K) = -\left(\frac{e}{m_e}\right)^2\left[\int dv_e\,\eta(v_e)F_1'(v_e)\,\chi(v_e),\,(v_e-u)^{-3}\right.$$
$$\left. - \int dv_e\,\eta'(v_e)F_1'(v_e)\,\chi(v_e)\,(v_e-u)^{-2}\right]$$
$$= -E \quad \text{(say)}. \tag{56}$$

Other relevant contributions are

$$\left.\begin{array}{c} R_{11}(K,2K) = 2A, \quad R_{21}(K,2K) = 2E, \\ R_{11}(2K,K) = \tfrac{1}{2}A, \quad R_{21}(2K,K) = \tfrac{1}{2}E, \\ R_{11}(K,K-K) = 0, \quad R_{21}(K,K-K) = 0, \\ T(K,K,\,\pm K) = 0, \end{array}\right\} \tag{57}$$

Substituting these into the expressions (41*b*) for $S(q,q',q'')$, we get

$$S(q,q',q'') = S_1(K) - i\,\mathrm{sgn}\,(q-q')\,S_2(K), \tag{58}$$

$$\begin{aligned} S_1 = {} & -\frac{(m_i A + m_e E)^2}{8K^2\,\epsilon(2K)} - \tfrac{1}{2}(m_i B + m_e D) + m_i C_1 + m_e C_3 \\ & + \{[(m_i\,\omega_1 + m_e\,\omega_3)^2 - (m_i\,\omega_2 + m_e\,\omega_4)^2]\,\Delta_1 \\ & + 2(m_i\,\omega_1 + m_e\,\omega_3)\,(m_i\,\omega_2 + m_e\,\omega_4)\,\Delta_2\}\,(\Delta_1^2 + \Delta_2^2)^{-1}, \end{aligned} \tag{59}$$

$$\begin{aligned} S_2 = {} & -(m_i C_2 + m_e C_4) - \{2(m_i\,\omega_1 + m_e\,\omega_3)\,(m_i\,\omega_2 + m_e\,\omega_4)\,\Delta_1 \\ & - [(m_i\,\omega_1 + m_e\,\omega_3)^2 - (m_i\,\omega_2 + m_e\,\omega_4)^2]\}\,. \end{aligned} \tag{60}$$

6. Derivation of the non-local nonlinear equation

We now proceed to derive the nonlinear equation after the corrections to the linear dispersion relation have been evaluated as above. Let us now consider the basic integral equation (27) and recall our basic assumption that the initial perturbation $f(K,v,t=0)$ is non-zero only for $K \approx \pm K_0$, that is, when ϕ is peaked around K_0. To make this explicit, we set

$$\hat{\phi}(K) = \psi(K-K_0) + \psi^*(-K-K^0), \tag{61}$$

with $\psi(q)$ non-zero only for $q \approx 0$. Then (27) becomes

$$\begin{aligned} -K^2\epsilon(p)\,\phi(p) = \phi_0(p) + i(2\pi)^{-2} & \int dq'\,dq'' \\ & \times S(q')\,\psi(q-q')\,\psi(q'-q'')\,\psi^*(-q'')\,[w-\Omega(K)]^{-1}, \end{aligned} \tag{62}$$

where we have changed the integration variable from q' to $q-q'$ so that

$$S(q') \equiv S(0,q-q',0) = S_1(K) - iS_2(K)\,\mathrm{sgn}\,(\mathrm{q}'). \tag{63}$$

Putting the same form of $\phi(p)$ on both sides of (62), that is,

$$\phi(p) = i\,[\psi(K-K_0) + \psi^*(-k-K_0)]\,[\omega-\Omega(K)]^{-1},$$

and considering the region of K near K_0, we arrive at

$$\begin{aligned} \epsilon(p)\,\phi(q) + (2\pi K)^{-2} & \int dq'\,dq'' \\ & \times S(q')\,\psi(q-q')\,\psi(q'-q'')\,\psi^*(-q'') = -i(\omega-\Omega)\frac{\phi_0}{K^2}. \end{aligned} \tag{64}$$

If the second term of this equation is proportional to $\psi(q)$ then the coefficient of $\psi(q)$ will give the frequency shift required for (17). This proportionality is most easily displayed through the Fourier transform with respect to q by treating $p = [K,\Omega(K)]$ as constant. For the left-hand side of (62), we then obtain

$$\left[\epsilon(p) + \int dx'\,S(x-x')\,|\,\psi(x')\,|^2/K^2\right]\psi(x),$$

which has the same form as (17). Therefore it is now possible to identify the kernel $Q(x)$ in the NLS equation:

$$Q(x) = \left[\delta_1\, \delta(x) + \frac{S_2}{\pi} p\left(\frac{1}{x}\right)\right], \tag{65}$$

whence the non-local nonlinear Schrödinger equation becomes

$$i\psi_t + p\psi_{xx} + Q_1 \psi |\psi|^2 + \frac{Q_2}{\pi}\psi p \int \frac{|\psi(x')|^2\, dx'}{x - x'} = 0, \tag{66}$$

where

$$Q_1 = \frac{S_1}{K^2}\left(\frac{\partial\epsilon}{\partial\omega}\right)^{-1}, \quad Q_2 = \frac{S_2}{K^2}\left(\frac{\partial\epsilon}{\partial\omega}\right)^{-1}, \tag{67}$$

$$K^2\left(\frac{\partial\epsilon}{\partial\omega}\right) = -e\frac{\partial}{\partial\omega}\left[\int\frac{F_2'(v_i)\, dv_i}{v_i - \omega/K} + \int\frac{\chi(v_e)\,\eta(v_e)\,F_1'(v_e)\, dv_e}{v_e - \omega/K}\right]. \tag{68}$$

7. The case of a Maxwellian plasma

Although it is known that the Maxwellian distribution does not apply in the relativistic situation, we shall still use it for the explicit estimation of the parameters in (66), since none of the integrals can be obtained exactly with a modified distribution function. Let us assume that the unperturbed velocity distributions for the electrons and ions are

$$\left.\begin{aligned} F_1(v_e) &= n(2\pi v_0^2)^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\frac{v_e}{v_0}\right)^2\right], \\ F_2(v_i) &= n(2\pi v_\alpha^2)^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\frac{v_i}{v_\alpha}\right)^2\right], \end{aligned}\right\} \tag{69}$$

where the electron thermal velocity

$$v_0 = \left(\frac{kT_e}{m_e}\right)^{\frac{1}{2}}$$

and the ion thermal velocity

$$v_\alpha = \left(\frac{kT_i}{m_i}\right)^{\frac{1}{2}}.$$

The relation between the frequency ω and wavenumber K turns out to be

$$\omega^2 = m_e\,\omega_{pe}^2 + m_i\,\omega p_i^2 + \frac{3K^2}{\omega^2}(m_e\,\omega_{pe}^2 v_0^2 + m_i\,\omega_{pi}^2 v_\alpha^2) + 15\frac{K^4}{\omega^4}(m_e\,\omega_{pe}^2 v_0^4 + m_i\,\omega_{pi}^2 v_\alpha^4), \tag{70}$$

where

$$\omega_{pe}^2 = \frac{4\pi_e^2 n}{m_e}, \quad \omega^2 p_i = \frac{4\pi e^2 n}{m_i}$$

are respectively the electron and ion plasma frequencies. Now we can evaluate the integrals with the help of (69), and we obtain

$$\omega_1(K) = -\left(\frac{e}{m_i}\right)^2 \frac{K^2}{\omega^2 v_g^2}\left(n+\frac{3nv_\alpha^2}{v_g^2}+\frac{15nv_\alpha^4}{v_g^4}\right), \tag{71a}$$

$$\omega_2(K) = \left(\frac{e}{m_i}\right)^2 \pi\frac{K^2}{\omega^2}F_2'(v_g), \tag{71b}$$

$$\omega_3(K) = \left(\frac{e}{m_i}\right)^2 n\left[\frac{K^2}{v_g^2\,\omega^2}\left(1+\frac{3v_0^2}{v_g^2}+\frac{15v_0^4}{v_g^4}\right) + \frac{3K}{C^2\omega v_g^2}\left(3v_0^2+\frac{15v_0^4}{v_g^2}-\frac{75v_0^6}{v_g^4}\right)\right], \tag{71c}$$

$$w_4(K) = \pi\left(\frac{e}{m_e}\right)^2\left(\frac{K^2}{\omega^2}+\frac{3K}{\omega}\frac{v_g}{C^2}\right)F_1'(v_g), \tag{71d}$$

$$C_1 = -\left(\frac{e}{m_i}\right)^3\frac{K^4}{\omega^4}\left(\frac{n}{v_g^2}+\frac{3nv_\alpha^2}{v_g^4}+\frac{15nv_\alpha^4}{v_g^6}\right), \tag{72a}$$

$$C_2 = -\left(\frac{e}{m_i}\right)^3\pi\frac{K^4}{\omega^4}F_2'(v_g), \tag{72b}$$

$$C_3 = -\left(\frac{e}{m_e}\right)^3 n\left[\frac{K^4}{\omega^4 v_g^2}+\frac{3K^4v_0^2}{\omega^4v_g^4}+\frac{15K^4v_0^4}{\omega^4v_g^6} + \frac{9K^3}{\omega^3C^2v_g^2}\left(\frac{25v_0^6}{v_g^4}-v_0^2-\frac{5v_0^4}{v_g^2}\right)+\frac{6K^3}{C^2\omega^3}\right], \tag{72c}$$

$$C_4 = -\left(\frac{e}{m_e}\right)^3\pi\frac{K^4}{\omega^4}F_1'(v_g), \tag{72d}$$

$$\Delta_1 = n\left(\frac{1}{v_g^2}+\frac{3v_\alpha^2}{v_g^4}+\frac{15v_\alpha^4}{v_g^6}+\frac{1}{v_g^2}+\frac{3v_0^2}{v_g^4}+\frac{15v_0^4}{v_g^4}\right), \tag{73a}$$

$$\Delta_2 = \pi[F_2'(v_g)+F_1'(v_g)]. \tag{73b}$$

Finally, $v_g = \partial\omega/\partial K$ can easily be computed from (70). With these expressions at hand, we now have explicit forms of the coefficients Q_1, Q_2 and p occurring in (66). These are rather long, and so are given in the appendix.

8. Nonlinear Landau damping

We now proceed to estimate the amount and variation of nonlinear Landau damping by an approximate analysis of (66). Let us consider a perturbed solution of (66) in the form

$$\psi = \rho^{\frac{1}{2}}\exp\left(i\int\frac{\sigma\,d\xi}{2p}\right) \tag{74}$$

where ρ and σ are real functions of ξ and τ. We set

$$\left.\begin{aligned} \rho &= \rho_0+\rho_1\,e^{i(K\xi-\Omega\tau)},\\ \sigma &= \sigma_0+\sigma_1\,e^{i(K\xi-\Omega\tau)},\end{aligned}\right\} \tag{75}$$

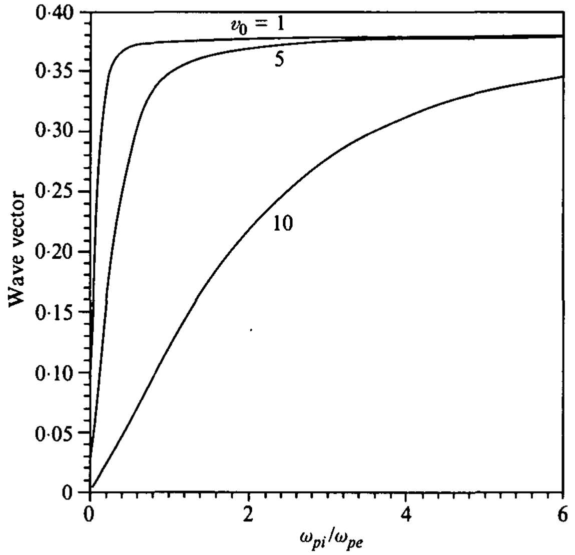

FIGURE 1. Wave vector given by the dispersion relation as a function of the frequency ratio ω_{pi}/ω_{pe} for $v_\alpha = 0{\cdot}5$, $\omega = 0{\cdot}5$ and different v_0.

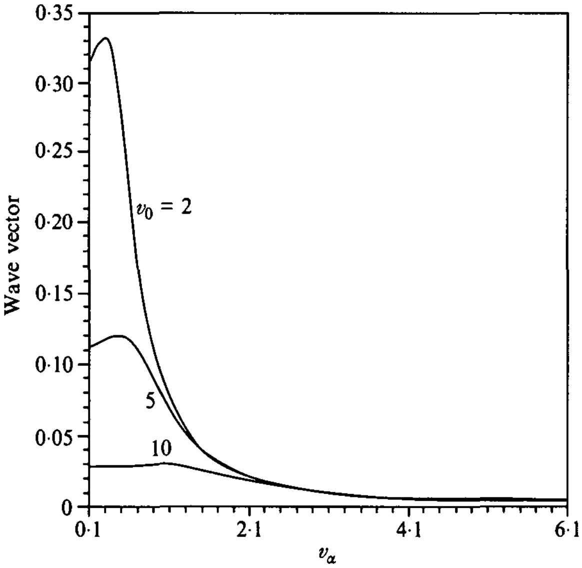

FIGURE 2. Wave vector given by the dispersion relation as a function of the frequency ratio v_α for $\omega_{pi}/\omega_{pe} = 2$; $\omega = 0{\cdot}5$ and different v_0.

where $\rho_1 \ll \rho_0$ and $\sigma_1 \ll \sigma_0$. Substituting into (66), we get

$$(\Omega - \sigma_0 K)^2 = -K^2 p \{2\rho_0 [Q_1 - i \operatorname{sgn}(K) Q_2 - k^2 p]\}. \tag{76}$$

Setting $\Omega = \Delta W + i\Gamma$, for $K > 0$ we get from (76)

$$\Gamma = \pm (\tfrac{1}{2}p)^{\frac{1}{2}} \{[(2\rho_0 Q_1 - pK^2)^2 + (2\rho_0 Q_2)^2]^{\frac{1}{2}} + (2\rho_0 Q_1 - pK^2)\}^{\frac{1}{2}} K. \tag{77}$$

which gives the nonlinear Landau damping as a function of several plasma parameters. To ascertain the behaviour of Γ, we start by analysing the

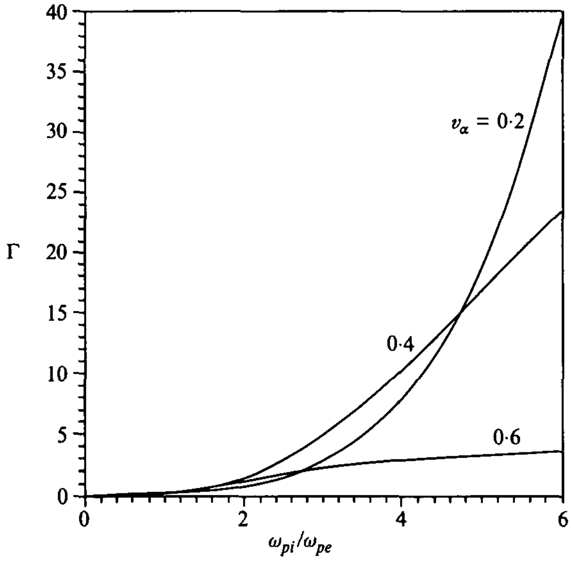

FIGURE 3. Nonlinear Landau-damping rate Γ as a function of frequency ratio ω_{pi}/ω_{pe} for $v_0/c = 0{\cdot}5$, $\omega = 0{\cdot}8$ and different v_α, exhibiting the tendency of growth for small v_α and large ω.

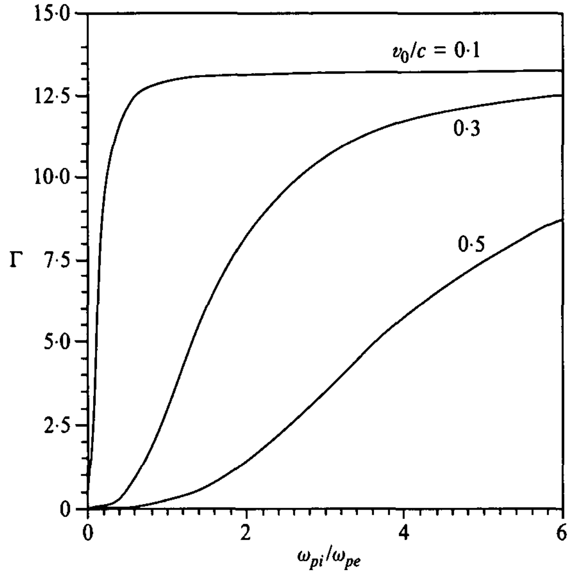

FIGURE 4. Nonlinear Landau-damping rate Γ as a function of frequency ratio ω_{pi}/ω_{pe} for $v_\alpha = 0{\cdot}5$, $\omega = 0{\cdot}5$ and different v_0/c, exhibiting the tendency of growth for large v_0/c and large ω.

dispersion relation (70) for several values of v_0, as a function of ω_{pi}/ω_{pe} and v_α. The corresponding solutions of the wave vector K are shown in figures 1 and 2. Equation (70) is then used to estimate the behaviour of the nonlinear Landau-damping coefficient Γ. This is clearly shown in figures 3–6. In figure 3 we observe that the Landau damping coefficient Γ grows quite steeply with ω_{pi}/ω_{pe} for fixed v_0/c and ω but different v_α. In figure 4 we show the variation of Γ for fixed v_α and

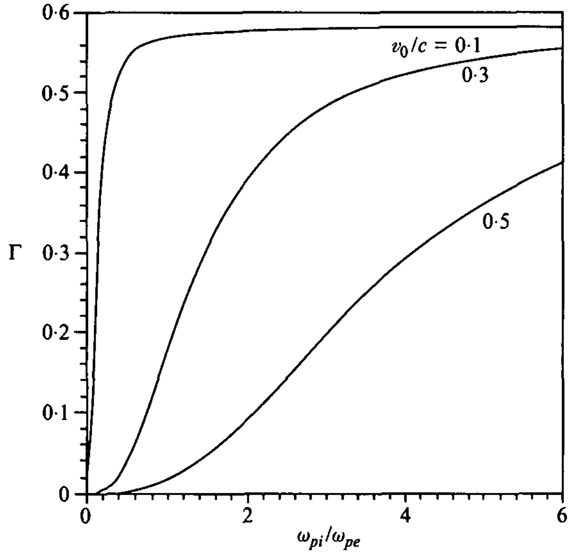

FIGURE 5. Nonlinear Landau-damping rate Γ as a function of frequency ratio ω_{pi}/ω_{pe} for $v_\alpha = 0·5$, $\omega = 0·5$ and different v_0/c, exhibiting the tendency of growth for large v_0/c.

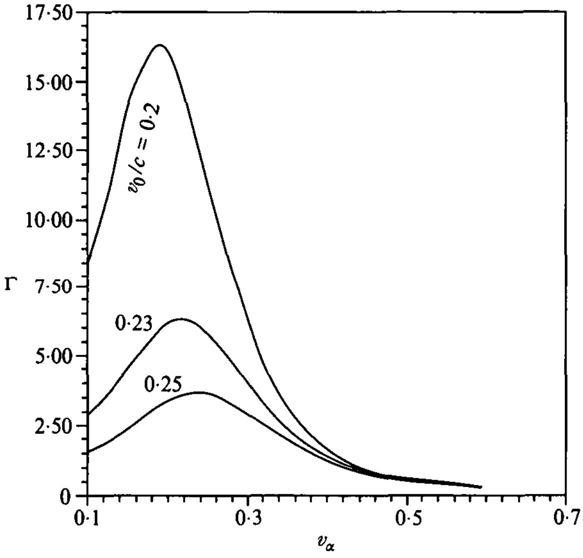

FIGURE 6. Nonlinear Landau Damping rate Γ as a function of the ion thermal velocity v_α for different v_0/c, but fixed ω_{pi}/ω_{pe}.

ω but different v_0/c. For low values of v_0/c the damping coefficient reaches a constant value, but for larger v_0/c it again increases sharply. On the other hand, in figure 5 the same variation is shown for lower frequencies. The qualitative behaviour remains the same, but the magnitude of Γ is scaled down by a factor of 20. In figure 6 the variation of Γ with respect to v_α is shown for fixed ω_{pi}/ω_{pe} and ω but different v_0/c. It is found that the larger the value of v_0/c, the lower is the value of Γ, but ultimately for large v_α they converge to same value, almost

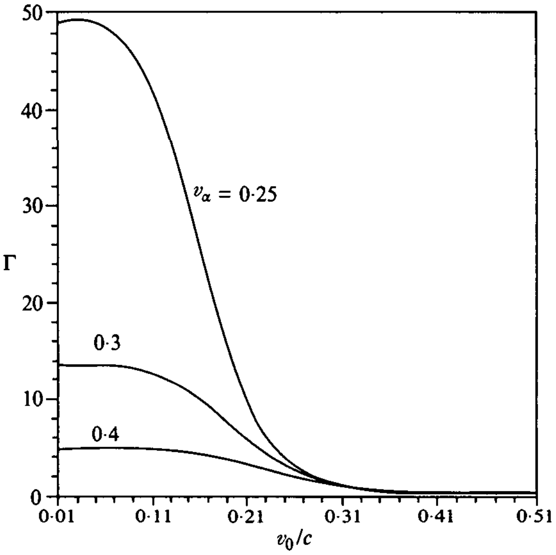

FIGURE 7. Nonlinear Landau-damping rate Γ as a function of v_0/c for $\omega_{pi}/\omega_{pe} = 2$, $\omega = 0\cdot5$ and different v_α.

zero. Finally, in figure 7 we have interchanged the role of v_α and v_0/c; that is, we have plotted Γ as a function of v_0/c for different v_α. The same phenomenas are seen to occur.

Thus, from the non-local nonlinear Schrödinger equation, we have obtained an estimate for the nonlinear Landau-damping coefficient for several values of the plasma parameters v_0/c, v_α, ω_{pi}/ω_{pe} and ω. Whether such an equation can sustain any soliton-like solution is yet to be seen.

9. Discussion

In the above analysis we have shown how the Vlasov plasma kinetic equations can be effectively used in conjunction with the approach of Fried & Ichikawa (1973) to derive a generalized non-local nonlinear Schrödinger equation. It may be noted that a similar equation has been derived previously for a non-relativistic plasma by a completely different method. In this respect, it may be mentioned that the actual velocity distribution for the relativistic case is not a Maxwellian one but rather is a Juttner function. However, none of the above integrals could be performed explicitly for such a distribution, so we have used the Maxwellian function to describe the plasma approximately. Another approach to the derivation of (66) utilizes a Fourier-mode expansion, which is more complicated from a computational point of view. The last part of our analysis has involved explicit computation of the nonlinear Landau-damping coefficient for various plasma parameters, all of which have been depicted graphically.

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Appendix

The explicit forms for Q_1, Q_2 and p in (66) are as follows:

$$
\begin{aligned}
Q_1 &= \frac{S_1}{K^2}\left(\frac{\partial \epsilon}{\partial \omega}\right)^{-1} \\
&= \frac{1}{K^2}\Big\{m_i c_1 + m_e c_3 + \{[(m_i \omega_1 + m_e \omega_3)^2 - (m_i \omega_2 + m_e \omega_4)^2]\,\Delta_1 \\
&\quad + 2(m_i \omega_1 + m_e \omega_3)(m_i \omega_2 + m_e \omega_4)\,\Delta_2\}\,(\Delta_1^2 + \Delta_2^2]\Big\} \\
&\quad \times \left[2m_e \omega_{pe}^2\left(\frac{1}{\omega^3} + \frac{6v_0^2 K^2}{\omega^5} + \frac{45 v_0^4 K^4}{\omega^7}\right)\right. \\
&\quad \left. + 2m_i \omega_{pi}^2\left(\frac{1}{\omega^3} + \frac{6v_\alpha^2 K^2}{\omega^5} + \frac{45 v_\alpha^4 K^4}{\omega^7}\right)\right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
Q_2 &= \frac{S_2}{K^2}\left(\frac{\partial \epsilon}{\partial \omega}\right)^{-1} \\
&= \frac{1}{K^2}\Big\{-(m_i C_2 + m_e C_4) - \{2(m_i \omega_1 + m_e \omega_3)(m_i \omega_2 + m_e \omega_4)\,\Delta_1 \\
&\quad - [(m_i \omega_1 + m_e \omega_3)^2 - (m_i \omega_2 + m_e \omega_4)^2]\}\Big\} \\
&\quad \times \left[2m_e \omega_{pe}^2\left(\frac{1}{\omega^3} + \frac{6v_0^2 K^2}{\omega^5} + \frac{45 v_0^4 K^4}{\omega^7}\right)\right. \\
&\quad \left. + 2m_i \omega_{pi}^2\left(\frac{1}{\omega^3} + \frac{6v_\alpha^2 K^2}{\omega^5} + \frac{45 v_\alpha^4 K^4}{\omega^7}\right)\right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
p &= \frac{1}{2}\frac{\partial^2 \omega}{\partial K^2} \\
&= \frac{B}{2[6\omega^5 - 4(m_e \omega_{pe}^2 + m_i \omega_{pi}^2)\,\omega^3]^2}
\end{aligned}
$$

where

$$
\begin{aligned}
B &= [6\omega^5 - 4(m_e \omega_{pe}^2 + m_i \omega_{pi}^2)\,\omega^3] \\
&\quad \times \left\{(m_e \omega_{pe}^2 v_0^2 + m_i \omega_{pi}^2 v_\alpha^2)\left[6K^2 + 12K\omega + (12k\omega + 6K^2)\frac{\partial \omega}{\partial K}\right]\right. \\
&\quad \left. + 180K^2(m_e \omega_{pe}^2 v_0^4 + m_i \omega_{pi}^2 v_\alpha^4)\right\} \\
&\quad - [(6k\omega^2 + 6K^2\omega)(m_e \omega_{pe}^2 v_0^2 + m_i \omega_{pi}^2 v_\alpha^2) + 60K^3(m_e \omega_{pe}^2 v_0^4 + m_i \omega_{pi}^2 v_\alpha^4)] \\
&\quad \times [30\omega^4 - 12\omega^2(m_e \omega_{pe}^2 + m_i \omega_{pi}^2)]\frac{\partial \omega}{\partial K},
\end{aligned}
$$

with

$$
\frac{\partial \omega}{\partial K} = \frac{6K\omega(\omega + K)(m_e \omega_{pe}^2 v_0^2 + m_i \omega_{pi}^2 v_\alpha^2) + 60K^3(m_e \omega_{pe}^2 v_0^4 + m_i \omega_{pi}^2 v_\alpha^4)}{6\omega^5 - 4(m_e \omega_{pe}^2 + m_i \omega_{pi}^2)\,\omega^3}.
$$

REFERENCES

BARYAKHTAR, V. G., CHESNOUSENKO, V. M., EROKHIN, N. S., SITENKO, A. G. & ZAKHAROV, V. E. (eds) 1992 *Plasma Theory and Nonlinear Process in Physics*. World Scientific.

CHAKRAVORTY, S., PAKIRA, G. & ROY CHOWDHURY, A. 1993 *Nuovo Cim.* **15***D*, 1043.

CHEN, F. F. 1984 *Introduction to Plasma Physics and Controlled Fusion*, 2nd edn, vol. I. Plenum.

DAVIDSON, R. C. 1992 *Methods in Nonlinear Plasma Theory*. Academic.

DEWAR, R. L. 1972 *J. Plasma Phys.* **7**, 267.

FRIED, B. D. & ICHIKAWA, Y. H. 1973 *J. Phys. Soc. Japan* **34**, 1073.

ICHIKAWA, Y., IMAMURA, T. & TANIUTI, T. 1972 *J. Phys. Soc. Japan* **33**, 189.

KADOMTSEV, B. B. 1965 *Plasma Turbulence*. Academic.

MJ0LHUS, E. & WYLLER, J. 1986 *Physica Scripta* **33**, 442.

OTT, E. & SUDAN, R. 1969 *Phys. Fluids* **13**, 143.

ROY CHOWDHURY, A., PAKIRA, G. & PAUL, S. N. 1988 *Physica* C **151**, 518.

ROY CHOWDHURY, K. & ROY CHOWDHURY, A. 1991 *Nuovo Cim.* **13***D*, 699.

SOM, B. K., GUPTA, M. R., DAS GUPTA, B. & JANAKI, S. 1992 *Physica Scripta* **45** 368.

WEILAND, J. 1981 *Eur. J. Phys.* **2**, 171.

WEILAND, J., ICHIKAWA, Y. H. & WILHELMSSON, H. 1977 *Nagoya Preprint* 1PPJ-315 (December).